LECTURE 7: FUN WITH CURVATURE

INTRODUCTION

We have already (informally) introduced the curvature $F(A)$ of a connection A. In gauge theories, the fundamental dynamical degrees of variables are given by a connection A, which appears in the Yang–Mills action

$$S(A) = \int_M \mathrm{Tr}(F(A) \wedge \star F(A)), \tag{1}$$

where $G = SU(N)$ and Tr is the trace in the fundamental representation, through the curvature. By contrast, in four dimensions, the similar looking

$$\int_M \mathrm{Tr}(F(A) \wedge F(A)),$$

does not lead to interesting dynamics at all: in fact it doesn't even depend on the connection A! This is an example of a *topological term* and the goal of this lecture is to explain why this is the case.

1. PRELIMINARIES

Let G be a fixed compact Lie group, for example $G = SU(N)$. Recall that we are working on a principal G-bundle $P \to M$ and that this bundle is completely determined by specifying an open covering $\{U_\alpha\}_{\alpha \in I}$, together with transition functions $\{\varphi_{\alpha\beta} : U_{\alpha\beta} \to G\}$ satisfying the cocycle identity

$$\varphi_{\alpha\gamma} = \varphi_{\alpha\beta}\varphi_{\beta\gamma},$$

on triple overlaps $U_{\alpha\beta\gamma}$. In terms of these data, the vector bundle $(P \times V)/G$ associated to a representation $\rho : G \to GL(V)$ is given by the same covering together with the transition functions $\{\rho \circ \varphi_{\alpha\beta}\}$. This gives a very concrete way to identity the bundles associated to certain tensors. For example, we have seen that the local connection A_α transforms on overlaps as

$$A_\alpha := \varphi_{\alpha\beta}^{-1} A_\beta \varphi_{\alpha\beta} + d\varphi_{\alpha\beta}\varphi_{\alpha\beta}^{-1}.$$

Therefore, a connection cannot be a section of some vector bundle: there is no vector bundle with transition functions involving the last term in the equation above. However, if we take the *difference* of two connections $A - B$, we see that these transform locally as

$$(A - B)_\beta = \varphi_{\alpha\beta}^{-1} A_\beta \varphi_{\alpha\beta},$$

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i.e., they are a section of the vector bundle associated to the adjoint representation: $\mathrm{ad}(P) := (P \times \mathfrak{g})/G$. We have already seen this before:

Lemma 1.1. *The difference between two connections $A - B \in \Omega^1(M, \mathrm{ad}(P))$.*

Another example is given by the curvature of a connection A defined by the equation

$$F(A) := dA + A \wedge A.$$

We have seen that, pulling back this form along local sections $s_\alpha : U_\alpha \to P$ gives $F_\alpha := s_\alpha^* F \in \Omega^2(U_\alpha, \mathfrak{g})$ satisfying

$$F_\alpha = \varphi_{\alpha\beta} F_\beta \varphi_{\alpha\beta}^{-1}.$$

We therefore conclude that there is a unique $F \in \Omega^2(M, \mathrm{ad}(P))$ such that $F(A) := \pi^* F$. We usually identify the curvature $F(A)$ with this $\mathrm{ad}(P)$-valued two form. This curvature form satisfies an important identity, for which we first observe that the connection A on P defines a connection on any of its associated vector bundles, given in the form of a covariant derivative, in particular the bundle associated to the adjoint representation:

Lemma 1.2 (Bianchi identity)**.**

$$\nabla_A^{\mathrm{Ad}(P)} F(A) = 0.$$

Proof. This is a local computation:

$$\begin{aligned}
\nabla_A^{\mathrm{Ad}(P)} F(A)|_{U_\alpha} &= d(dA_\alpha + A_\alpha \wedge A_\alpha) + [A_\alpha, dA_\alpha + A_\alpha \wedge A_\alpha] \\
&= dA_\alpha \wedge A_\alpha - A_\alpha \wedge dA_\alpha + A_\alpha \wedge dA_\alpha - dA_\alpha \wedge A_\alpha \\
&= 0.
\end{aligned}$$

□

Remark 1.3 ($A \wedge A$ or $\frac{1}{2}[A, A]$?)**.** For matrix Lie groups we write $A \wedge A$ which means combining the $\wedge$-product with matrix multiplication:

$$(A \wedge A)(X, Y) = \frac{1}{2}(A(X)A(Y) - A(Y)A(X)),$$

where X and Y are vector fields on P. Now the right hand side has a meaning for any Lie algebra valued 1-form: we therefore can also write $\frac{1}{2}[A, A]$.

2. The Chern–Weil homomorphism

Let $P \to M$ be a principal G-bundle, and denote by $\mathfrak{g}$ the Lie algebra of G.

Definition 2.1. An *invariant homogeneous polynomial* is given by a polynomial map $P : \mathfrak{g} \to \mathbb{C}$ which is invariant under the adjoint action of G:

$$P(\mathrm{Ad}_g(X)) = P(X), \quad \text{for all } X \in \mathfrak{g},\ g \in G.$$

We denote the algebra of invariant polynomials of arbitrary degree by $I_{\mathrm{inv}}(G)$.

We denote by $I^k_{\mathrm{inv}}(G)$ the invariant polynomials of degree k. Closely related to an invariant polynomials are invariant symmetric multilinear maps

$$\tilde{P} : \underbrace{\mathfrak{g} \times \ldots \times \mathfrak{g}}_{k \text{ times}} \to \mathbb{C}$$

invariant under the action of G:

$$\tilde{P}(\mathrm{Ad}_g(X_1), \ldots, \mathrm{Ad}_g(X_k)) = \tilde{P}(X_1, \ldots, X_k), \quad \text{for } X_i \in \mathfrak{g} \text{ and } g \in G.$$

The correspondence between P and $\tilde{P}$ is as follows: given $\tilde{P}$, define

$$P(X) := \tilde{P}(X, \ldots, X),$$

the restriction to the diagonal. Conversely, given P, define its *polarization* as

$$\tilde{P}(X_1, \ldots, X_k) := \frac{(-1)^k}{k!} \sum_{j=1}^{k} \sum_{i_1 < \ldots < i_j} P(X_{i_1} + \ldots + X_{i_j}).$$

For example, for $k = 2$ we have

$$\tilde{P}(X_1, X_2) := \frac{1}{2}(P(X_1 + X_2) - P(X_1) - P(X_2)).$$

In the following we will therefore refer to both P as well as $\tilde{P}$ interchangeably as an invariant polynomial.

Given a principal G-bundle $P \to M$ and $P \in I^k_{\mathrm{inv}}(G)$, we pick a connection A on P with curvature $F(A) \in \Omega^2(M, \mathrm{ad}(P))$ and consider the differential form

$$P(F(A)) \in \Omega^{2k}(M). \tag{2}$$

To make sense of this expression, choose, for a point $x \in M$ an isomorphism $\mathrm{ad}(P)_x \cong \mathfrak{g}$ of the fiber of $\mathrm{ad}(P)$. Then we can apply P to $F(A) \in \Omega^2(M, \mathrm{ad}(P))$ at that point. Since P is invariant, its value is in fact independent of the chosen isomorphism, and combining with the wedge product, this yields a smooth differential form of degree $2k$.

Theorem 2.2 (Chern–Weil)**.**

i) *The form $P(F(A))$ is closed: $dP(F(A)) = 0$.*
ii) *The induced cohomology class in $H^{2k}_{\mathrm{dR}}(M)$ is independent of the chosen connection A.*

Proof. First remark that by invariance of P we have

$$\sum_{i=1}^{k} P(X_1, \ldots, [A, X_i], \ldots, X_k) = 0, \quad A, X_1, \ldots, X_k \in \mathfrak{g}.$$

This identity can be obtained by using invariance with respect to conjugation with $g = \exp(tA)$ and differentiation. Therefore, in a local trivialization where we write A_α for the connection 1-form, we find

$$\begin{aligned} dP(F_\alpha,\dots,F_\alpha) &= \sum_{i=1}^{k} P(F_\alpha,\dots,dF_\alpha,\dots,F_\alpha) \\ &= \sum_{i=1}^{k} P(F_\alpha,\dots,\nabla^{\mathrm{ad}(P)}F_\alpha - [A_\alpha,F_\alpha],\dots,F_\alpha) \\ &= 0 \end{aligned}$$

by the Bianchi identity and the invariance of P. This proves the first claim.

For the second, let A' be another connection. By Lemma 1.1 we have $A' = A + \alpha$ for some $\alpha \in \Omega^1(M;\mathrm{End}(E))$. Therefore the convex combination $A_t = tA' + (1-t)A = A + t\alpha,\ t \in [0,1]$ is a family of connections interpolating between A and A'. We now consider A_t as a connection on the principal bundle $P \times [0,1] \to M \times [0,1]$.

A small computation shows that

$$F(A_t) = F(A) + dt \wedge \alpha + t\nabla\alpha + t^2\alpha \wedge \alpha \in \Omega^2(M \times [0,1], \mathfrak{g}).$$

For an invariant polynomial P of degree k, we now consider the fiber integral over t-parameter:

$$L(\nabla,\nabla') := \int_0^1 P(F(A_t)) \in \Omega^{2k-1}(M).$$

(To evaluate this integral, we pick the terms in $P(F(A_t))$ which contain one factor dt and then perform the integral.) This L is called the *transgression form*. Stokes' theorem now gives:

$$\begin{aligned} dL(\nabla,\nabla') &= d\int_0^1 P(F(A_t)) \\ &= \int_0^1 dP(F(A_t)) - P(F(A_t))|_{t=1} + P(F(A_t))|_{t=0} \\ &= P(F(A)) - P(F(A')). \end{aligned}$$

This proves the second claim. □

Corollary 2.3 (Chern–Weil homomorphism)**.** *Given a principal bundle $P \to M$, there is a canonical homomorphism of graded algebras*

$$I_{\mathrm{inv}}(G) \to H^{2\bullet}_{\mathrm{dR}}(M).$$

3. Chern classes of vector bundles

In the previous section we have defined characteristic classes of principal bundles. To define such cohomology classes for *vector bundles*, we could go over to the frame bundle of a vector bundle: this is a principal $GL(r,\mathbb{C})$ (for a complex vector bundle of rank r), to which the theory of the previous section applies, and for any invariant polynomial P on the Lie algebra $\mathrm{Mat}_r(\mathbb{C})$ of $r \times r$-matrices, we obtain a characteristic class.

Chern classes of vector bundles are examples associated to specific examples of invariant polynomials. Before we discuss these, let us briefly explain how to compute directly the curvature out of a connection (i.e., a covariant derivative) on a vector bundle.

3.1. **Curvature on vector bundles.** Let $E \to M$ be a vector bundle equipped with a connection ∇. Using the Leibniz identity, we can extend a connection to an operator $\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M; E)$ by

$$\nabla(s \otimes \alpha) = \nabla s \wedge \alpha + s \otimes d\alpha, \quad s \in \Gamma^\infty(M, E),\ \alpha \in \Omega^k(M).$$

The operator ∇ thus defined doesn't turn $\Omega^\bullet(M; E)$ into a complex: $\nabla^2 \neq 0$. However we do have

$$\nabla^2(fs) = f\nabla^2(s), \quad \text{for all } f \in C^\infty(M)$$

so we can define the curvature $F(\nabla) \in \Omega^2(M, \mathrm{End}(E))$ by

$$F(\nabla)(s) := \nabla^2(s) \in \Omega^2(M; \mathrm{End}(E)) \quad \text{for all } s \in \Gamma^\infty(M, E).$$

In a local trivialization $\nabla|_{U_\alpha} = d + A_\alpha$ we see that

$$F_\alpha := F(d + A_\alpha) = (d + A_\alpha)^2 = dA_\alpha + A_\alpha \wedge A_\alpha. \tag{3}$$

Indeed with this we see that under the local gauge transformation (**??**) we have

$$\begin{aligned}
F_\alpha &= d(\varphi_{\alpha\beta}A_\beta\varphi_{\alpha\beta}^{-1} - (d\varphi_{\alpha\beta})\varphi_{\alpha\beta}^{-1}) + (\varphi_{\alpha\beta}A_\beta\varphi_{\alpha\beta}^{-1} - (d\varphi_{\alpha\beta})\varphi_{\alpha\beta}^{-1}) \wedge (\varphi_{\alpha\beta}A_\beta\varphi_{\alpha\beta}^{-1} - (d\varphi_{\alpha\beta})\varphi_{\alpha\beta}^{-1}) \\
&= \varphi_{\alpha\beta}dA_\beta\varphi_{\alpha\beta}^{-1} + \varphi_{\alpha\beta}(A_\beta \wedge A_\beta)\varphi_{\alpha\beta}^{-1} \\
&= \varphi_{\alpha\beta}F_\beta\varphi_{\alpha\beta}^{-1}.
\end{aligned}$$

This is precisely the transformation property of a section of the bundle $\mathrm{End}(E) \to M$ in local trivializations. It is remarkable that, although a connection is not a section of any bundle associated to E, the curvature does have this property. From the local expression for the curvature above, together with Eq. (**??**), we can deduce the useful formula

$$F(\nabla)(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in \mathfrak{X}(M).$$

Lemma 3.1 (Bianchi identity). *The curvature of a connection satisfies:*

$$\nabla^{\mathrm{End}(E)}(F(\nabla^E)) = 0.$$

Proof. Write out:

$$\begin{aligned}
\nabla^{\mathrm{End}(E)}(F(\nabla^E))(s) &= \nabla^E(F(\nabla)(s)) - F(\nabla^E)(\nabla^E(s)) \\
&= (\nabla^E)^3(s) - (\nabla^E)^3(s) \\
&= 0.
\end{aligned}$$

Here we have used the definition (**??**) of the connection $\nabla^{\mathrm{End}(E)}$ on the bundle $\mathrm{End}(E)$ induced by ∇^E. $\square$

3.2. **Chern classes.** Consider now the invariant polynomials $P_k \in I^k_{\mathrm{inv}}$ defined by

$$\det(I + tA) =: P_0(A) + tP_1(A) + t^2 P_k(A) + \dots, \quad A \in \mathrm{Mat}_r(\mathbb{C}).$$

(Of course, $\det(1 + tgAg^{-1}) = \det(1 + tA)$, so the polynomials are indeed invariant.) The polynomial P_k defines the *k-th Chern class*[1]

$$c_k(E) := \left(\frac{\sqrt{-1}}{2\pi}\right)^k P_k(F(\nabla)) \in H^{2k}_{\mathrm{dR}}(X).$$

For example, using the well-known expansion

$$\det(I + tA) = r + t\mathrm{Tr}(A) + \frac{t^2}{2}\left(\mathrm{Tr}(A^2) - \mathrm{Tr}(A)^2\right) + \dots + t^r \det(A),$$

we find in low degrees

$$\begin{aligned}
c_0(E) &= \mathrm{rank}(E) \in H^0_{\mathrm{dR}}(M),\\
c_1(E) &= \frac{\sqrt{-1}}{2\pi}\mathrm{Tr}(F(\nabla)) \in H^2_{\mathrm{dR}}(M),\\
c_2(E) &= -\frac{1}{4\pi^2}\left(\mathrm{Tr}(F(\nabla) \wedge F(\nabla)) - \mathrm{Tr}(F(\nabla)) \wedge \mathrm{Tr}(F(\nabla))\right) \in H^4_{\mathrm{dR}}(M).
\end{aligned}$$

The *total Chern class* is defined as

$$c(E) := \sum_{k \geq 0} c_k(E).$$

Proposition 3.2. *The total chern class $c(E) \in H^\bullet_{\mathrm{dR}}(M)$ satisfies the following properties:*

i) (Naturality) for $f : N \to M$ a smooth map, we have

$$c(f^*E) = f^*c(E) \in H^\bullet_{\mathrm{dR}}(N),$$

ii) (Product formula) For a direct sum $E \oplus F$, we have

$$c(E \oplus F) = c(E)c(F)$$

The first property follows from the fact that the pull-back connection $f^*\nabla$ on f^*E has curvature equal to $F(f^*\nabla) = f^*F(\nabla)$. The second property follows from the fact that the direct sum connection $\nabla^E \oplus \nabla^F$ on $E \oplus F$ has curvature that can be written in matrix form as

$$\begin{pmatrix} F(\nabla^E) & 0 \\ 0 & F(\nabla^F) \end{pmatrix}.$$

In general, Chern classes measure how "nontrivial" a vector bundle is. To witness this point, we have:

Lemma 3.3. *For a trivializable vector bundle $E \to M$, all Chern classes $c_k(E)$, $k \geq 1$ are zero.*

[1]The reason for the normalization factor $\frac{\sqrt{-1}}{2\pi}$ is the fact that with precisely this factor the Chern classes are *integtral*, c.f. Theorem **??**

Proof. Let us first remark that on the trivial vector bundle $M \times \mathbb{C}^r$ we can choose the trivial connection given by the exterior derivative d applied to vector-valued functions. This connection has curvature zero since $d \circ d = 0$ and therefore the trivial bundle has vanishing Chern classes. For a trivializable vector bundles, assume that $\varphi : E \xrightarrow{\cong} M \times \mathbb{C}^r$ is a trivialization. Then E carries a connection given by

$$\nabla = \varphi^{-1} \circ d \circ \varphi = d + \varphi^{-1} d\varphi.$$

We have already seen that under such "gauge transformations" φ, the curvature transforms neatly:

$$F(\varphi^{-1} \circ d \circ \varphi) = \varphi F(d) \varphi^{-1} = 0,$$

so again the Chern classes are zero. Notice that the theory implies that any other connection ∇ on E, its *Chern forms* $c_k(E, \nabla) \in \Omega^{2k}(M)$ are exact. □

Finally, we come a crucial property of Chern classes, namely that they are *integral* cohomology classes:

Theorem 3.4. *Chern classes of complex vector bundles are* integral*. This means that for any complex vector bundle vector bundle $E \to M$ and any closed compact $2k$-dimensional oriented submanifold $S \subset M$, the integral*

$$\int_S c_k(E)$$

is an integer.

These numbers are called *Chern numbers*. Notice that the fact that the differential form $c_k(E) \in \Omega^{2k}(M)$ is closed explains that the value of the integral does not depend on the precise embedding of S into M, in fact by de Rham's theorem only the underling *homology class* in $H_{2k}^{\mathrm{sing}}(M, \mathbb{R})$ (obtained by taking the fundamental class) matters. However, the fact that the value of these integrals are always integers is truly remarkable.